A NOTE ON BOSONIZATION OF 1 DIMENSIONAL ELECTRON GAS

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Abstract. We study the original development of bosonization chronologically, and investigate in the one dimensional electron gas - Luttinger liquid. Many important results are reproduced with help from various sources.

1. A brief introduction of the historical development

More than two hundred years ago, a French mathematician Augustin-Louis Cauchy found the identity which named after him:

$$\det \frac{1}{z_i - w_j} = \frac{\prod_{i<j}(z_i - z_j)(w_i - w_j)}{\prod_{i,j}(z_i - w_j)}. \tag{1}$$

In the 1970s it was interpreted in the context of two-dimensional (2D) quantum field theory. Suppose we have a chiral fermionic Lagrangian, $\mathcal{L} = \frac{i}{2\pi}\ \psi_R^\dagger \partial_z \psi_R$, where $\partial_z = \frac{1}{2}\ (\partial_x + \partial_t)$. Then, as it follows from Wick's theorem, the l.h.s. of eq.(1) coincides with 2n-point correlation functions $\langle \psi_R(z_1)\cdots\psi_R(z_N)\psi_R^\dagger(w_1)\cdots\psi_R^\dagger(w_N)\rangle$.

At the same time its r.h.s. can be understood as the correlation function of the exponential fields built from the chiral boson $\langle e^{i\phi_R(z_1)}\ldots e^{i\phi_R(z_N)}e^{-\phi_R(w_1)}\ldots e^{-i\phi_R(w_N)}\rangle$.

In fact, Cauchy's identity suggests the remarkable relation $\psi_R(z) = e^{i\phi_R(z)}$ between the Fermi and Bose fields. All the above can be repeated to the left movers and formulated as an equivalence between non-chiral free Fermi and Bose theories described by the Lagrangians[1]

$$\mathcal{L}_{\mathrm{F}} = i\bar{\Psi}\partial\!\!\!/\Psi \quad \sim \quad \mathcal{L}_{\mathrm{B}} = \tfrac{1}{2}\left(\partial_\mu \Phi\right)^2 . \tag{2}$$

In the seminal work [1] in 1975, Coleman extended this equivalence and "bosonized" the interacting theory:

$$\mathcal{L}_{\mathrm{F}}^{(g,M)} = \bar{\Psi} i\partial\!\!\!/\Psi - \tfrac{g}{2}\ \bar{\Psi}\gamma^\mu\Psi\bar{\Psi}\gamma_\mu\Psi + M\bar{\Psi}\Psi, \tag{3}$$

by means of the so-called sine-Gordon model

$$\mathcal{L}_{SG} = \tfrac{1}{2}\ (\partial_\mu\Phi)^2 + \cos\Phi \tag{4}$$

Among others the bosonization implies that $U(1)$ current in the fermionic theory coincides with the topological current in the sine-Gordon model:

$$\bar{\Psi}\gamma^\mu\Psi = -\frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\Phi\ . \tag{5}$$

[1] Here $\Psi = \frac{1}{\sqrt{2\pi}}\left(\psi_R, \psi_L\right)^T$, whereas $\Phi = 2\sqrt{\pi}(\phi_R + \phi_L)$.

The above identification between the boson operator and fermion bilinear indicates that boson is a bound state of fermion-antifermion pair. It starts from the comparison of the correlation functions of fermionic fields and exponential of bosonic fields. He found them are equal if the coupling constants satisfy certain condition. This leads to the identity of currents in (5). In the same year of 1975, explicit formulae for single fermion field were obtained by Mandelstam [2] :

$$\psi_{1,2} = C_{1,2}\, e^{2\pi i \int_{-\infty}^{x} d\zeta\, \dot{\phi}(\zeta) \mp \frac{1}{2} i\phi(x)}\ . \tag{6}$$

Historically, there is a parallel development of bosonization in condensed matter physics. In the 1950s Tomonaga [5] first identified the boson-behaved density operator for interacting fermions, his analysis was based on the manipulation of the boson-behaved fermionic bilinear operator $\psi^\dagger\psi$. Later the bosonization identity $\psi(x) \sim e^{i\phi(x)}$ was used by various studies in 1D electron systems.

Luttinger model is a model which describes interacting electrons in 1D, and it is the massless version of eqn (3). The motivation to study such system theoretically is that it actually describes systems in the reality. Although our world is 3D, there exists systems that can be described effectively by 1D models, such as quantum wires. Bosonization technique can be introduced to study such model with interaction at low energy excitation ($\epsilon \sim \epsilon_F$), where linear dispersion relation can be assumed. Recently it appears that nonlinear Luttinger liquid is under investigation. The most interesting phenomenon of Luttinger liquid is spin charge separation, where spin and charge density are treated as separated particles propagating independent of each other.

In this note, we will first discuss the original discovery of Bosonization identity by Coleman. Then we will bosonize the xzz spin chain for the non interactive case in continuum limit. Finally we will go through constructive bosonization that is frequently used in condensed matter physics. In the last section, we will briefly motion some experimental phenomenon of Luttinger liquid.

2. Coleman's identity

In this section we derive explicitly the n point correlation function of the free boson field, and make comparison to the n point correlation function of the free fermion field. The identities in Coleman's paper [1] follow directly from the comparison of the correlation functions.

2.1. **2 point correlation function.**

The euclidean action for free boson is, by Francesco (2.96) [10]:

$$S = \frac{1}{2}\int d^2x\, \partial_\mu\phi\partial^\mu\phi + m^2\phi^2$$

As worked out in Srednicki [11] chapter 8, the propagator is $\frac{1}{k^2+m^2}$ (in Euclidean space). And in 2 dimension we can integrate it out:

$$\Delta(x) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{ikx}}{k^2+m^2} = \int \frac{k\,dkd\phi}{(2\pi)^2} \frac{e^{i|k||x|\cos\phi}}{k^2+m^2}$$
$$= \frac{1}{2\pi} \int dk \frac{kJ_0(|k||x|)}{k^2+m^2} = \frac{K_0(m|x|)}{2\pi}$$

Here we used the identity for the Bessel functions: $\int_0^{2\pi} d\phi e^{ix\cos\phi} = 2\pi J_0(x)$ and $\int_0^\infty dx \frac{xJ_0(x)}{x^2+m^2} = K_0(m)$. Expand around $x = 0$, $K_0(m|x|) = -\gamma + \ln 2 - \ln mx$. So that $\Delta(x) = -\frac{1}{2\pi} \ln cmx$.

2.2. **N point corollation function.**

2.2.1. *Normal ordering method.* From Coleman (4.2), he first considered a correlation function

$$T < 0, \mu| \prod_i N_m e^{i\beta_i \phi(x_i)} |0, \mu >$$

Knowing the form of free propagator of boson field is $\Delta_F = -\frac{1}{4\pi} \ln c\mu^2 x^2$. He concluded that with $N_m e^{i\beta\phi}$ replaced by $(\frac{\mu^2}{m^2})^{\beta^2/8\pi} e^{i\beta\phi}$ through the identity (2.11), we could have the result of the correlation function:

$$(\frac{\mu^2}{m^2})^{(\sum \beta_i^2/8\pi)} \prod_{i>j} [c\mu^2(x_i - x_j)^2]^{\beta_i\beta_j/4\pi}$$

Ignore the constant in front of the exponential operators, we want to show the result of $< 0|\Pi_i : e^{X_i} : |0 >$, especially $< 0| \prod_i : e^{t_i X_i} : |0 >$. The colon is used to indicate normal ordering. Let us begin with:

$$: x^{X_1} : \cdots : e^{X_n} : \tag{7}$$

where $X_i = X_i^+ + X_i^-$, plus and minus X_i are associated with creating and annilation operators, repectively. Using the Weyl's identity in the third equality:

$$: e^X :=: e^{X^+ + X^-} := e^{X^+} e^{X^-} = e^{\frac{1}{2}[X^+, X^-]} e^{X^+ + X^-} = e^{\frac{1}{2}[X^+, X^-]} e^X$$

Then 7 becomes:

$$\prod_i e^{-\frac{1}{2}[X_i^-, X_i^+]} e^{X_1} \cdots e^{X_n} \tag{8}$$

Applying the general Weyl's identity on 8, we have:

$$
\begin{aligned}
&\prod_i e^{-\frac{1}{2}[X_i^-,X_i^+]}\prod_{i<j} e^{\frac{1}{2}[X_i,X_j]}e^{X_1+\cdots+X_n} \\
=&\prod_i e^{-\frac{1}{2}[X_i^-,X_i^+]}e^{X_1}\prod_{i<j} e^{\frac{1}{2}[X_i,X_j]}e^{(X_1^++\cdots+X_n^+)+(X_1^-+\cdots+X_n^-)} \\
=&\prod_i e^{-\frac{1}{2}[X_i^-,X_i^+]}e^{X_1}\prod_{i<j} e^{\frac{1}{2}[X_i,X_j]}e^{(X_1^++\cdots+X_n^+)+(X_1^-+\cdots+X_n^-)} \\
=&\prod_i e^{-\frac{1}{2}[X_i^-,X_i^+]}e^{X_1}\prod_{i<j} e^{\frac{1}{2}[X_i,X_j]}e^{\frac{1}{2}[X_1^-+\cdots+X_n^-,X_1^++\cdots+X_n^+]} : e^{X_1+\cdots+X_n} : \\
=&\prod_{i<j} e^{[X_i^-,X_j^+]} : e^{X_1+\cdots+X_n} :
\end{aligned}
\tag{9}
$$

Therefore:

$$
\begin{aligned}
&< 0| : x^{t_1X_1} : \cdots : e^{t_nX_n} : |0> \\
=&< 0|\prod_{i<j} e^{t_it_j[X_i^-,X_j^+]} : e^{t_1X_1+\cdots+t_nX_n} : |0> \\
=&\prod_{i<j} e^{t_it_j[X_i^-,X_j^+]} < 0| : e^{t_1X_1+\cdots+t_nX_n} : |0> \\
=&\prod_{i<j} e^{t_it_j[X_i^-,X_j^+]} \\
=&\prod_{i<j} e^{t_it_j<0|X_i,X_j|0>}
\end{aligned}
\tag{10}
$$

Then Coleman (4.5) [1] follows immediately, and (4.11) as a more specific case can be checked using (4.5) with small n.

2.2.2. *Path integral method.* When making calculation, we can add a source $J\phi$ in the Lagrangian, and take $J \to 0$ after the calculation. The path integral can be written, as shown in Srednicki chapter 7 and 9, and also Francesco (2.107):

$$\int D\phi\, e^{-\frac{1}{2}\int d^2x[\partial^\mu\phi\partial_\mu\phi+m^2\phi^2-J(x)\phi(x)]} = e^{\frac{1}{2}\int d^2x\, d^2y\, J(x)\Delta(x-y)J(y)}$$

Therefore the n point exponential corollation function can be written as:

$$T < 0|\prod_i e^{i\beta_i\phi(x_i)}|0 >= \prod_i e^{i\beta_i\delta_i}e^{\frac{1}{2}\int d^2x\, d^2y\, J(x)\Delta(x-y)J(y)}$$

Where $\delta_i J(x) = \delta^2(x_i - x)$. Taking $J \to 0$, we immediately get

$$
\begin{aligned}
& e^{\sum_{i\neq j} -\frac{1}{2}\Delta(x_i - x_j)\beta_i\beta_j} \\
=& e^{\sum_{i\neq j} \frac{\beta_i\beta_j}{8\pi} \ln cm^2(x_i-x_j)^2} \\
=& e^{\sum_{i<j} \frac{\beta_i\beta_j}{4\pi} \ln cm^2(x_i-x_j)^2} \\
=& \prod_{i<j} [cm^2(x_i - x_j)^2]^{\frac{\beta_i\beta_j}{4\pi}}
\end{aligned}
$$

This is in agreement with Coleman (4.5). Now we can specialize our computation to get the result of Coleman (4.11).

$$
\begin{aligned}
& T\,\langle \prod_i e^{i\beta(\phi(x_i)-\phi(y_i))} \rangle \\
=& \prod_i e^{i\beta(\delta_{x_i}-\delta_{y_i})} e^{\frac{1}{2}\int d^2x d^2y J(x)\Delta(x-y)J(y)} \\
=& e^{\sum_{i\neq j} -\frac{1}{2}\beta^2\Delta(x_i-x_j)} e^{\sum_{i\neq j} -\frac{1}{2}\beta^2\Delta(y_i-y_j)} e^{\sum_{i,j} \frac{1}{2}\beta^2\Delta(x_i-y_j)} \\
=& \frac{\prod_{i<j}[(x_i-x_j)^2m^2c]^{\frac{\beta^2}{4\pi}} \prod_{i<j}[(y_i-y_j)^2m^2c]^{\frac{\beta^2}{4\pi}}}{\prod_{i,j}[(x_i-y_i)m]^{\frac{\beta^2}{4\pi}}}
\end{aligned}
\tag{11}
$$

This is in agreement with Coleman's result (4.11).

2.3. **Partition function of Sine Gordon model of neutral boson gas.**

The partition function of the Sine-Gordon model can be written as

$$
Z = \int D\phi\, e^{\int d^2x -\frac{1}{2}(\partial^\mu\phi)^2 + 2\zeta\cos\beta\phi}
$$

Using the result from previous section, we can calculate it:

$$
Z = \int D\phi\, e^{-\frac{1}{2}\int d^2x\,(\partial_\mu\phi)^2} \sum_{n=0}^{\infty} \int d^2x_1 \cdots d^2x_n\, \frac{\zeta^n}{n!}(e^{i\beta\phi(x_1)} + e^{-i\beta\phi(x_1)}) \cdots (e^{i\beta\phi(x_n)} + e^{-i\beta\phi(x_n)})
$$

Only even n could survive, after we take $J \to 0$. Thus we can let $n = 2m$, where m is a non-negative integer. Since the gas is neutral, there are $C(2m, m)$ ways of combining charges, we need to attach this factor in the partition function:

$$
\begin{aligned}
Z &= \int D\phi\, e^{-\frac{1}{2}\int d^2x\,(\partial_\mu\phi)^2} \sum_{q=\pm 1}\sum_{m=0}^{\infty}\int d^2x_1\cdots d^2x_{2m}\frac{\zeta^{2m}}{(2m)!}\frac{(2m)!}{(m!)^2}\prod_{i=1}^{2m} e^{i\beta q_i\phi(x_i)} \\
&= \sum_{q_i=\pm 1}\sum_{m=0}^{\infty}\int d^2x_1\cdots d^2x_{2m}\frac{\zeta^{2m}}{(m!)^2}\langle\prod_{i=1}^{2m} e^{i\beta q_i\phi(x_i)}\rangle \\
&= \sum_{q_i=\pm 1}\sum_{m=0}^{\infty}\int d^2x_1\cdots d^2x_{2m}\frac{\zeta^{2m}}{(m!)^2} e^{\sum_{i\neq j}-\frac{1}{2}\beta^2 q_i q_j\Delta(x_i-x_j)}
\end{aligned}
$$

2.4. **Massless free Dirac fermions.** The massless Thirring model has Laguangian:

$$\mathcal{L} = \bar{\psi} i\gamma_\mu\partial^\mu\psi - \frac{1}{2}gj^\mu j_\mu$$

where $j^\mu = \bar{\psi}\gamma^\mu\psi$. Since the massless Thirring model is soluble, we can shift the Laguangian by $-m'\sigma$, where $\sigma = Z\bar{\psi}\psi$, and m' is just a real parameter and has nothing to do with mass. As indicated by Coleman, the calculation of vacuum expectation values of arbitrary strings of ψ and $\psi^\dagger$ is lengthy. The result can be obtained from Klaiber:

$$\langle T\prod_{i=1}^{n}\sigma_+(x_i)\sigma_-(y_i)\rangle = (\frac{1}{2})^{2n}\frac{\prod_{i>j}[(x_i-x_j)^2(y_i-y_j^2)M^4]^{(1+b/\pi)}}{\prod_{i,j}[M^2(x_i-y_j)^2]^{(1+b/\pi)}} \tag{12}$$

where $b = -\frac{g}{1+g/\pi}$ is just Klaiber's choice of parameter.

2.5. **Equivalence between the two models.** Now compare 11 and 12, we can see they are equal if $M^2 = cm^2$, $-\sigma_\pm = \frac{1}{2}A_\pm$ and $\frac{1}{1+g/\pi} = \frac{\beta^2}{4\pi}$. Since the perturbation term in Laguangin is $\frac{\alpha}{2\beta^2}(A_+ + A_-)$ for Sine-Gordon model and $m'(\sigma_+ + \sigma_-)$ for Thirring model, the two perturbation theories are the same if $m' = \alpha/\beta$. Therefore, we can identity the boson field ϕ in the Sine-Gordon model with the bilinear $\bar{\psi}\psi$ in the massless Thirring model as

$$-\bar{\psi}\psi \propto \cos\beta\phi$$

For convenience, lets write down the Lagrangian for both theories here again:

$$
\begin{aligned}
\mathcal{L}_{SG} &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\alpha_0}{\beta^2}\cos\beta\phi - \frac{\alpha_0}{\beta^2} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\alpha_0}{2\beta^2}(A_+ + A_-) - \frac{\alpha_0}{\beta^2} \\
\mathcal{L}_{TH} &= i\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}gj^\mu j_\mu - m'Z\bar{\psi}\psi = i\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}gj^\mu j_\mu - m'(\sigma_+ + \sigma_-)
\end{aligned}
$$

As shown in Coleman, not only can we identify the perturbative terms Lagrangians, we can also identify the current in Thirring model as the derivative term in Sine-Gordon model. The commutation relations

$$
\begin{aligned}
&[j^\mu(x), \sigma_\pm(y)] = \mp 2(1+\frac{g}{\pi})^{-1}\epsilon^{\mu\nu}D_\nu(x-y)\sigma_\pm(y) \\
&[\partial_\nu\phi(x), A_\pm(y)] = \pm\beta D_\nu(x-y)A_\pm(y)
\end{aligned}
\tag{13}
$$

are identical if $j^\mu = -\frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi$.

3. Bosonization of massless fermion theory

The first example of bosonization in field theory started from writing the Heisenberg spin chain in the form of spinless fermions by Jordan Wigner transformation [6]. Such spinless fermionic system can be mapped to the sine-Gordon theory by Luther and Peschel [7] in 1975. In the following subsections, we follow the spirit of such motivation, making the Heisenberg model a effective boson model with two subsequent transformations.

3.1. **Heisenberg spin chain in fermionic form.** We would like to represent the Heisenberg spin system by a massless fermion system. Define the raising and lowering operators at each site of spin chin n, $S^\pm(n)$:

$$S^\pm(n) = S_1(n) \pm iS_2(n), \quad S_i = \frac{1}{2}\sigma_i \tag{14}$$

These operators commute on different sites for $m \neq n$, and they anticommute on the same site:

$$\begin{aligned} &[S^+(n), S^+(m)] = [S^-(n), S^-(m)] = [S^+(n), S^-(m)] = 0 \\ &\{S^+(n), S^-(n)\} = I \\ &\{S^+(n), S^+(n)\} = \{S^-(n), S^-(n)\} = 0 \end{aligned} \tag{15}$$

With such condition the operator creates bosonic excitations on different sites, and there cannot be two excitations on the same site. Define kink operator that rotate sites before n around the z-axis:

$$K(n) = e^{(i\pi\sum_{j=1}^{n-1} S_j^+ S_j^-)} \tag{16}$$

Define the fermion operator (Jordan-Wigner transformation) as flippling a spin followed by creating a kink:

$$c(n) = K(n)S^-(n) \tag{17}$$

The inverse of Jordan-Wigner transformation is:

$$S^-(n) = e^{-i\pi\sum_{n=1}^{n-1} c^\dagger(j)c(j)}c(n) \tag{18}$$

These fermion operators satisfy the anticommutation relation, since $S^- * S^- = 0$. It can be shown (try it out on up and down states) that $\{S_n^+, K_n\} = 0$, $[S_n^-, K_m] = 0, \forall n \neq m$. One more identity that relates S_3 to S^+ and S^- is:

$$\begin{aligned} c^\dagger(n)c(n) &= S^+(n)S^-(n) = \frac{1}{2} + S_3(n) \\ c(n)c^\dagger(n) &= S^-(n)S^+(n) = \frac{1}{2} - S_3(n) \end{aligned} \tag{19}$$

Hence all the spin operators have the form in terms of raising and lowering operators. The Heisenberg chain can be written as:

$$\begin{aligned} H &= \sum_i J(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + \gamma J(S_i^z S_{i+1}^z) \\ &= \sum_j J(S_j^\dagger S_{j+1}^- + S_j^- S_{j+1}^+) + \gamma J(S_j^+ S_j^- - \frac{1}{2})(S_{j+1}^+ S_{j+1}^- - \frac{1}{2}) \end{aligned} \tag{20}$$

Applying the inverse Jordan-Wigner transformation, the spin Hamiltonian becomes fermionic:

$$H = \frac{J}{2}\sum_j^N (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + \gamma J \sum_{j=1}^N (n_j - \frac{1}{2})(n_{j+1} - \frac{1}{2}) \tag{21}$$

Setting γ to zero yields the X-Y model:

$$H_0 = \frac{J}{2}\sum_{j=1}^N (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) = \int_k J\cos(ka) c_k^\dagger c_k, \quad \text{where } c_j = \int \frac{dk}{2\pi} e^{ikx_j} c_k$$

This system is gapless and the ground is obtained by filling up negative energy states.

3.2. **Continuum limit of spin chain.** However, the spin chain is a discrete system, and we have to make it to a continuum limit so that fermionic representation becomes possible.

3.2.1. *Non-interacting case.* We consider:

$$H_0 = \frac{J}{2}\sum_{n=1}^N (c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n), \quad \epsilon(k) = J\cos(ka) \tag{22}$$

We should expect correlation functions to have a factor of e^{ikx}. Since k has to be around the fermi surface, $e^{ik_F n}$. For half filled case, $|k_F| = \frac{\pi}{2} \Rightarrow e^{ik_F n} = i^n$. This factor varies as we move along the lattice, hence it is neccessary to remove this fluctuation:

$$a_n = i^{-n} c_n \tag{23}$$

The new Hamiltonian becomes:

$$H_0 = \frac{J}{2}\sum_{s=1}^{N/2} i\{a_{2s}^\dagger (a_{2s+1} - a_{2s-1}) + a_{2s+1}^\dagger (a_{2s+2} - a_{2s})\} \tag{24}$$

Define the spinor fields $\phi_1(n) = a_{2s}$ for even n and $\phi_2(n) = a_{2s+1}$ for odd n. Then the Hamiltonian in new spinor fields is:

$$H_0 = \frac{J}{2}\sum_{s=1}^{N/2} i\{\phi_1^\dagger(2s)[\phi_2(2s+1) - \phi_2(2s-1)] + \phi_2^\dagger(2s+1)[\phi_1(2s+2) - \phi_1(2s)]\} \tag{25}$$

Since the spinor ϕ field is defined by operator c on discrete lattice, its anticommutation relation is discrete:

$$\{\phi_\alpha^\dagger(n), \phi_{\alpha'}(n')\} = \delta_{\alpha\alpha'}\delta_{nn'} \tag{26}$$

For a conanical fermion field $\psi_\alpha(x)$ on a continuum, we expect a continous version of anticommutation relation:

$$\{\psi^\dagger_\alpha(x), \psi_{\alpha'}(x')\} = \delta_{\alpha\alpha'}\delta(x - x') \tag{27}$$

This can be achieved by defining:

$$\psi_\alpha(x) = \frac{1}{\sqrt{2a}}\phi_\alpha(n), \quad \delta(x - x_0) = \lim_{a\to 0}\frac{\delta_{nn'}}{2a} \tag{28}$$

By such definition, the difference of discrete fields can be written as the derivative of continuous field:

$$\begin{aligned}\phi_2(2s+1) - \phi_2(2s-1) &= \sqrt{2a}\,2a\frac{\partial}{\partial x}\psi_2(x)\\ \phi_1(2s+2) - \phi_1(2s) &= \sqrt{2a}\,2a\frac{\partial}{\partial x}\psi_1(x)\end{aligned} \tag{29}$$

Also, by changing the summation to integration, $\lim_{a\to 0}\sum_s 2af(s) = \int dx f(x)$:

$$\begin{aligned}H_0 &= \frac{J}{2}i\sum_s\{\psi^\dagger_1(x)(2a)^2\partial_x\psi_2(x) + \psi^\dagger_2(2a)^2\partial_x\psi_1(x)\}\\ &= iJa\int dx\{\psi^\dagger_1\partial_x\psi_2 + \psi^\dagger_2\partial_x\psi_1\}\\ &= iJa\int dx(\psi^\dagger_1, \psi^\dagger_2)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\partial_x\psi_1\\ \partial_x\psi_2\end{pmatrix} = Jai\int dx\psi^\dagger\sigma_1\partial_x\psi\end{aligned} \tag{30}$$

Define $\widetilde{H_0} := \frac{H_0}{Ja}$, hence we find the Hamiltonian for spinor field:

$$\widetilde{H_0} = \int dx\psi^\dagger\sigma_1 i\partial_x\psi \tag{31}$$

To put these into chirl fermion field representation, where σ_1 becomes diagonal. We can define:

$$\begin{aligned}\psi_1(x) &= \frac{1}{\sqrt{2}}(-R(x) + L(x))\\ \psi_2(x) &= \frac{1}{\sqrt{2}}(R(x) + L(x))\end{aligned} \tag{32}$$

Then:

$$\begin{aligned}&\psi^\dagger_1 i\partial_x\psi_2 + \psi^\dagger_2 i\partial_x\partial_1 = -(R^\dagger i\partial_x R + L^\dagger i\partial_x L) \Rightarrow\\ &\widetilde{H_0} = \int(L^\dagger, R^\dagger)i\sigma_3\begin{pmatrix}\partial_x L\\ \partial_x R\end{pmatrix} = \int\psi^\dagger i\gamma_0\gamma_1\psi = \int dx\bar{\psi}i\gamma_1\partial_x\psi\end{aligned} \tag{33}$$

3.2.2. *Interacting case.* Including the interaction term of Heisenberg Hamiltonian:

$$
\begin{aligned}
H_{int} &= -\frac{\gamma J}{2}\sum_{j=1}^{N}(a_j^\dagger a_j - a_{j+1}^\dagger a_{j+1})^2 + \frac{1}{4}\gamma JN \\
&= -\frac{1}{2}\gamma J\sum_{s=1}^{N/2}\{(\phi_1^\dagger(2s)\phi_1(2s) - \phi_2^\dagger(2s+1)\phi_2(2s+1))^2 + (\phi_2^\dagger(2s+1)\phi_2(2s+1) - \phi_1^\dagger(2s+2)\phi_1(2s+2)
\end{aligned}
$$

In the second expression of third line, the irrelevant constant $\frac{1}{4}\gamma JN$ is dropped. Taking the continuum limit:

$$
\begin{aligned}
H_{int} &= -\frac{1}{2}\gamma J\sum_{s=1}^{N/2}\{[\psi_1^\dagger\psi_1 - \psi_2^\dagger\psi_2]^2 + [\psi_2^\dagger\psi_2 - \psi_1^\dagger\psi_1]^2\}(2a)^2 \\
&= -\gamma Ja2\int dx[-\psi_1^\dagger\psi_1 + \psi_2^\dagger\psi_2]^2 \qquad (34)
\end{aligned}
$$

Working in chiral basis:

$$
\psi_2^\dagger\psi_2 - \psi_1^\dagger\psi_1 = R^\dagger L + L^\dagger R = (L^\dagger, R^\dagger)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}L\\ R\end{pmatrix} = \bar{\psi}\psi \qquad (35)
$$

The interacting Hamiltonian in chiral basis is:

$$
\widetilde{H}_{int} = -2\gamma\int dx(\bar{\psi}\psi)^2 = \gamma\int dx j_\mu j^\mu, \quad \text{where } j_\mu = \bar{\psi}\gamma_\mu\psi \qquad (36)
$$

It satisfies the chiral symmetry.

3.3. **Bosonization.** Now we complete the first step to convert the Heisenberg spin chain to an interactive fermion system. There are a lot of ways to study such system, bosonization is a good non-perturbative method. The $U(1)$ current is $j^\mu\bar{\psi}\gamma^\mu\psi$, in chiral basis it is $j_0 = L^\dagger L + R^\dagger R$ and $j_1 = L^\dagger L - R^\dagger R$. If the density and current operators are normalized, we obtain a Schwinger term following the calculation from [8],

$$
[j_0(x), j_1(x')] = -\frac{i}{\pi}\partial_x\delta(x - x') \qquad (37)
$$

However, for boson field we have such relation:

$$
[\phi(x), \Pi(x')] = i\delta(x - x') \qquad (38)
$$

This implies that there is a connection between fermion and boson fields, as current j^μ is bilinear of ψ.

$$
j_0(x) = \frac{1}{\sqrt{\pi}}\partial_x\phi, \quad j_1(x) = -\frac{1}{\sqrt{\pi}}\partial_t\phi(x) := -\frac{1}{\sqrt{\pi}}\Pi(x)
$$

Since both the left and right mover are conserved, there are two conserved currents. In this case it is the axial current:

$$
j_\mu^5 = \bar{\psi}\gamma_\mu\gamma^5\psi = \bar{\psi}\epsilon_{\mu\nu}\gamma^\nu\psi \Rightarrow j_\mu^5 = \epsilon_{\mu\nu}j^\nu \qquad (39)
$$

The above shows Coleman's identification between fermion bilinear and single boson field. To obtain the explicit bosonization of single fermion field by Mandelstam [2], consider two operators defined as:

$$
\begin{aligned}
O_\alpha(x) &= e^{i\alpha\phi(x)} \\
O_\beta(x) &= e^{i\beta\int_{-\infty}^{x_1} dx_1' \partial_0\phi(x_0,x_1')} = e^{i\beta\int_{-\infty}^{x_1} dx_1' \Pi(x_0,x_1')}
\end{aligned}
\tag{40}
$$

When acting on $|\{\phi(x)\}\rangle$, O_α simply attaches a phase factor of $e^{i\alpha\phi}$, whereas O_β shifts $\phi(x_0, x_1')$ to $\phi(x_0, x_1') + \beta, \forall x_1' < x$. Consider the operator $\psi_{\alpha\beta}(x)$ of the form:

$$
\psi_{\alpha,\beta}(x) = O_\alpha(x)O_\beta(x) = e^{i\alpha\phi(x)+i\beta\int_{-\infty}^{x_1} dx_1'\partial_0\phi(x_0,x_1')}
\tag{41}
$$

To see how this fermion operator satisfy the anticommutation relation, we multiply two of them and apply Baker-Hausduff rule $e^Ae^B = e^Be^Ae^{-[A,B]}$:

$$
\psi_{\alpha,\beta}(x)\psi_{\alpha,\beta}(x') = \psi_{\alpha,\beta}(x')\psi_{\alpha,\beta}(x)e^{-i\Phi(x,x')} = \psi_{\alpha,\beta}(x')\psi_{\alpha,\beta}(x)e^{i\alpha\beta}
\tag{42}
$$

Where the second and last expression come from:

$$
\begin{aligned}
i\Phi(x,x') = -\alpha^2[\phi(x),\phi(x')] - \beta^2\int_{-\infty}^{x_1} dy_1 \int_{-\infty}^{x_1'} dy_1' [\Pi(y),\Pi(y')] \\
-\alpha\beta\int_{-\infty}^{x_1'} dy_1'[\phi(x),\Pi(y')] - \alpha\beta\int_{-\infty}^{x_1} dy_1[\Pi(y),\phi(x')] = -i\alpha\beta
\end{aligned}
\tag{43}
$$

Clearly, we need $\alpha\beta = \pm\pi \Rightarrow \alpha = \pm\frac{\pi}{\beta}$ to have the correct anticommutation relation for fermion field. Mandelstam made separate expressions for left and right fermion fields:

$$
\begin{aligned}
R(x) &= \frac{1}{\sqrt{2\pi a}} : e^{-i\frac{2\pi}{\beta}\int_{-\infty}^{x_1'} dx_1'\Pi(x_0,x_1') + i\frac{\beta}{2}\phi(x)} : \\
L(x) &= \frac{1}{\sqrt{2\pi a}} : e^{-i\frac{2\pi}{\beta}\int_{-\infty}^{x_1'} dx_1'\Pi(x_0,x_1') - i\frac{\beta}{2}\phi(x)} :
\end{aligned}
\tag{44}
$$

The $U(1)$ current j^μ [2] is:

$$
j_\mu = \frac{\beta}{2\pi}\epsilon_{\mu\nu}\partial^\nu\phi
\tag{45}
$$

Compare the above current with the Coleman identity, we require that $\beta = \sqrt{4\pi}$. However, we should expect that the bosonized chiral fermion field could be represented by chiral boson field, it is true if we write boson field in mode expansion and separate it into chiral components:

$$\phi(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\frac{1}{2|k|}(a(k)e^{i(|k|x_0 - kx_1)} + a^\dagger(k)e^{-i(|k|x_0-kx_1)}) \Rightarrow$$

$$\Pi(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\frac{1}{2|k|}(i|k|a(k)e^{i(|k|a(k)x_0 - kx_1)} - i|k|a^\dagger(k)e^{-i(|k|x_0-kx_1)}) \Rightarrow$$

$$\phi_R(x) = \phi_R(x_0 - x_1) = \int_0^{\infty} \frac{dk}{2\pi}\frac{1}{2k}(a(k)e^{ik(x_0-x_1)} + a^\dagger(k)e^{-ik(x_0-x_1)})$$

$$\phi_L(x) = \phi_R(x_0 + x_1) = \int_0^{\infty} \frac{dk}{2\pi}\frac{1}{2k}(-a(k)e^{-ik(x_0+x_1)} - a^\dagger(k)e^{ik(x_0+x_1)}) \tag{46}$$

Introduce the duel field θ such that the field operator ϕ and it obey the Cauchy-Riemann equation:

$$\partial_0\phi = \Pi := \partial_1\theta \tag{47}$$

The fields finally decompose as chiral components as:

$$\phi(x_0, x_1) = \phi_R(x_0 - x_1) + \phi_L(x_0 + x_1)$$

$$\theta(x_0, x_1) = -\phi_R(x_0 - x_1) + \phi_L(x_0 + x_1) \tag{48}$$

Put the above expression into Mandelstam's identity, the chiral fermion is identical to the exponential of the chiral boson field:

$$R(x) = \frac{1}{\sqrt{2\pi a}} : e^{i2\sqrt{\pi}\phi_R} :$$

$$L(x) = \frac{1}{\sqrt{2\pi a}} : e^{-i2\sqrt{\pi}\phi_L} : \tag{49}$$

It is shown [8] that by explicit calculation, the operator product yields:

$$\lim_{y_1\to x_1} R^\dagger(x_0, x_1)L(x_0, y_1) = \frac{1}{2\pi a} : e^{-i2\sqrt{\pi}\phi(x)} :$$

$$\lim_{y_1\to x_1} L^\dagger(x_0, x_1)R(x_0, y_1) = \frac{1}{2\pi a} : e^{+i2\sqrt{\pi}\phi(x)} :$$

Now we have formulae to bosonize any fermion field that appears in Lagrangian, and then we put them together:

$$L_F = H_0 - H_{int} = i\bar{\psi}\partial\!\!\!/\psi - \gamma(\bar{\psi}\gamma_\mu\psi)^2 \tag{50}$$

$$= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\gamma}{\pi}\epsilon_{\mu\nu}\partial^\nu\phi\epsilon^{\mu\lambda}\partial_\lambda\phi \tag{51}$$

The first term of the last expression comes from the conservation of axial current j^5:

$$\partial_\mu j^{5\mu} = \epsilon^{\mu\nu}\partial_\mu j_\nu = \frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\epsilon_{\nu\lambda}\partial_\mu\partial^\lambda\phi = \frac{1}{\sqrt{\pi}}\partial^2\phi$$

$$\partial_\mu j^{5\mu} = 0 \Rightarrow \frac{1}{\sqrt{\pi}}\partial^2\phi = 0, \quad \partial^2 = \partial_0^2 - \partial_1^2 \tag{52}$$

If we rescale the boson field and coupling constant:

$$\varphi := \sqrt{1 + \frac{2\gamma}{\pi}}\phi \tag{53}$$

The bosonized Lagrangian becomes:

$$L_B = \frac{1}{2}(\partial_\mu \varphi)^2 \tag{54}$$

We now understand that the massless fermion theory is equivalent to free boson theory. Moreover, the interacting massive fermion system is equivalent to sine-Gordon model.

4. Luttinger model

Luttinger liquid is a general model of 1D electron gas at low energy regime. The shape of 1D system can be thought as a rectangle, with its length L way longer than its width d. By such assumption, the momentum is dense, because $\Delta p \sim \frac{2\pi}{L}$, and the gap between different energy states is large: $\Delta E \sim \frac{1}{d}$. Thus we only need to think it is a one band system with all electrons filling up to the fermi surface. To the nature of 1D system, the excited particles' momentum are either parallel or antiparallel to the direction of fermi energy (posivie for negative). This indicates that the excited particle and the hole it lefts behind are correlated at long distance - they form a long-lived bound state (collective mode). Therefore, one would suspect that the low energy excitation of fermions is effective theory of boson.

The retarded Green's function recognizes the probability of excitation for given set of energy and momentum. In Landau's fermi gas theory, the spectral function of free fermion is maximized but smears out for a wide range of momentum. However it is quite different for Luttinger liquid, where the spectral function has two poles at qv_f and $-qv_f$. This tells us the excitation consists of left and right movers as bosons.

4.1. **Spinless fermion.** Consider the Hamiltonian:

$$H = H_0 + H_{int} = \psi^\dagger(x)(\alpha i v_F \partial_x)\psi(x) + 2g_2\rho_R(x)\rho_L(x) + g_4(\rho_R^2(x) + \rho_L^2(x)) \tag{55}$$

where g_2 measures the strength of backscattering and g_4 that of the forward scattering. At precisely half-filling, the Umklapp interaction has to be incoorperated into the Hamiltonian. This allows scattering process to be momentum conserved up to a one reciproval lattice vector:

$$H_{Umklapp} = g_3 \lim_{y\to x}(\psi_R^\dagger(x)\psi_R^\dagger(y)\psi_L(x)\psi_L(y) + R \leftrightarrow L) \tag{56}$$

The Umklapp term cannot be written as the product of density operators, hence it breaks the continuous chiral symmetry of the Hamiltonian. Using the identities derived before, the free fermion Hamiltonian is:

$H_0 = \psi^\dagger(x)(\alpha i v_F \partial_x)\psi(x) = \frac{v_F}{2}(\Pi^2 + (\partial_x\phi)^2) = \frac{v_F}{2}((\partial_x\theta)^2 + (\partial_x\phi)^2)$ together with previous identities:

$\rho_R = \frac{1}{2\sqrt{\pi}}(\partial_x\phi - \pi) = \frac{1}{2\sqrt{\pi}}\partial_x(\phi - \theta), \quad \rho_L = \frac{1}{2\sqrt{\pi}}(\partial_x\phi + \pi) = \frac{1}{2\sqrt{\pi}}\partial_x(\phi + \theta)$

The new Hamiltonian without Umklapp term is:

$$H = (\pi v_F + g_4)(\rho_R^2 + \rho_L^2) + 2g_2\rho_R\rho_L = \frac{v}{2}(\frac{1}{K}(\partial_x\theta)^2 + K(\partial_x\phi)^2) \quad \text{where} \tag{57}$$
$$v = \sqrt{(v_F + \frac{g_4}{\pi})^2 - (\frac{g_2}{\pi})^2}, \quad K = \sqrt{\frac{v_F + g_4/\pi + g_2/\pi}{v_F + g_4/\pi - g_2/\pi}}$$

The bosonized Hamiltonian is invariant with duality transformation:

$$\phi \leftrightarrow \theta, \quad K \leftrightarrow \frac{1}{K} \tag{58}$$

The forward scattering factor g_4 renormalizes the speed and the backward scattering factor g_2 renormalizes the stiffness constant K. Clearly, for repulsive interaction $g_2 > 0$, $K > 0$; and for attractive interaction $g_2 < 0$, $K < 0$. For the case of half filling that includes Umklapp scattering term, the bosonized Hamiltonian has an additional term $g_\mu \cos(4\pi\phi)$ that breaks the chiral symmetry of the system.

4.2. **Spin fermion.** Consider the spin Hamiltonian:

$$\begin{aligned} H = -iv_F \sum_{\sigma=\uparrow,\downarrow}\sum_{s=\pm} s\psi^\dagger_{s,\sigma}\partial_x\psi_{s,\sigma} + g_4\sum_{\sigma,s}\psi^\dagger_{s,\sigma}\psi^\dagger_{s,-\sigma}\psi_{s,-\sigma}\psi_{s,\sigma} + g_2\sum_{\sigma,\sigma'}\psi^\dagger_{1,\sigma}\psi^\dagger_{-1,\sigma'}\psi_{-1,\sigma'}\psi_{1,\sigma} \\ + g_{1,\|}\sum_\sigma \psi^\dagger_{1,\sigma}\psi^\dagger_{-1,\sigma}\psi_{1,\sigma}\psi_{-1,\sigma} + g_{1,\perp}\sum_\sigma \psi^\dagger_{1,\sigma}\psi^\dagger_{-1,-\sigma}\psi_{1,-\sigma}\psi_{-1,\sigma} \end{aligned} \tag{59}$$

The first line has the kinetic term, the forward scattering of the same branch and opposite branch. The second line includes the backscattering process. Also the Umklappp term should be considered at half filling case:

$$H_u = g_3 e^{i(4p_F - G)x}\psi^\dagger_{-1,\uparrow}\psi^\dagger_{-1,\downarrow}\psi_{1,\downarrow}\psi_{1,\uparrow} + h.c. \tag{60}$$

The H_0 term can be easilly expressed as:

$$H_0 = \frac{v_F}{2}\sum_\sigma(\Pi_\sigma^2 + (\partial_x\phi_\sigma)^2) \tag{61}$$

Define the charge and spin boson fields and transform the free Hamiltonian:

$$\begin{aligned} \phi_c = \frac{1}{\sqrt{2}}(\phi_\uparrow + \phi_\downarrow), \quad \phi_s = \frac{1}{\sqrt{2}}(\phi_\uparrow - \phi_\downarrow) \Rightarrow \\ H_0 = \frac{v_F}{2}(\pi_c^2 + (\partial_x\phi_c)^2) + \frac{v_F}{2}(\Pi_s^2 + (\partial_x\phi_s)^2) \end{aligned} \tag{62}$$

The bosonized free Hamiltonian describes the charge and spin fields moving at the same velocity. Adding interaction terms will renormalize the speed of the propagating bosonic waves:

(63)

$$H = \frac{v_c}{2}(\frac{1}{K_c}\Pi_c^2 + K_c(\partial_x\phi_c)^2) + \frac{v_s}{2}(\frac{1}{K_s}\Pi_s^2 + K_s(\partial_x\phi_s)^2) + V_c\cos(2\sqrt{2\pi}\phi_c) + V_s\cos(2\sqrt{2\pi}\phi_s)$$

$$v_c = \sqrt{(v_F + \frac{g_4}{2\pi})^2 - (\frac{g_{1,\parallel}}{2\pi} - \frac{g_2}{\pi})^2}, \quad v_s = \sqrt{(v_F - \frac{g_4}{2\pi})^2 - (\frac{g_{1,\parallel}}{2\pi})^2}$$

$$K_c = \sqrt{\frac{2\pi v_F + g_4 + 2g_2 - g_{1,\parallel}}{2\pi v_F + g_4 - 2g_2 + g_{1,\parallel}}}, \quad K_2 = \sqrt{\frac{2\pi v_F - g_4 - g_{1,\parallel}}{2\pi v_F - g_4 + g_{1,\parallel}}}$$

$$V_c = \frac{g_3}{2(\pi a)^2}, \quad V_s = \frac{g_{1,\perp}}{2(\pi a)^2}$$

Clearly the ineractions modify the spin and charge velocity, thus it leads to the spin charge separation.

4.3. **Scaling.** Bosonization is a method that maps a system to another system, it has no use to solve the problem. We only hope the mapped system is easier to analysis so we could see the behavior of the original system that is hard to study. For the sine-Gordon model, we would like to find the limit that the interacting cosine term is useful, in other word if it is an important term adding to the free boson Langrangian.

Suppose we are interested in the correlation function of operator $\phi_n(r)$.If the system is at fixed point such that it is translation, rotation and scale invariant; we would expect the correlation function have a power law dependence on the distance:

$$\langle\phi_n(r)\phi_n(r')\rangle = \frac{1}{|r - r'|^{2\Delta_n}} \tag{64}$$

where Δ_n is the scaling dimension of ϕ_n. For sine-Gordon model, we are interested in the correlation of $\cos(\varphi) = \frac{1}{2}(e^{i\varphi} + e^{-i\varphi})$, the vertex operator. For sine-Gordon Langrangian in Euclidean space of the generic form $L = \frac{K}{2}(\partial\varphi)^2 + \frac{u}{a^2}\cos\varphi$, the correlation of vertex operator is:

$$\langle e^{in\varphi(0)}e^{-in\varphi(x)}\rangle = (\frac{1}{z})^{n^2/(4\pi K)}(\frac{1}{\bar{z}})^{n^2/(4\pi K)} \tag{65}$$

Thus the scaling dimension of the vertex operator is $\Delta_1 = \frac{n^2}{4\pi K} = \frac{n^2\beta^2}{4\pi}$. A similar analysis shows that the scaling dimension of $(\partial\varphi)^2$ is 2, which is independent of any parameter. Thus if $\Delta_1 > 2$, the interaction is irrelavant, marginal if $\Delta_1 = 2$, and relavant if $\Delta_1 < 2$.

For Luttinger model, we may write it as the summation of the charge and spin sections $H = H_c + H_s$. Simillar to the above analysis, $K_c = 1 \Rightarrow \Delta_c = 2$, the Umklapp process is marginal. $K_c > 1$ makes it relevant, and it is associated with the repulsive interaction. For spin section, the scaling dimension behaves exactly the same with respect to K_s.

5. Bosonization identity from condensed matter physics

We may view bosonization as a tool that can be constructed from few simple assumptions of physics system. This method of building bosonization identity is called constructive bosonization. The starting point only requires the definition of fermion field in terms of fermion creation and annihilation operators which statisfy the usual anticommutation relation:

$$\psi_\eta(x) = (\frac{2\pi}{L})^{1/2} \sum_k e^{-ikx} c_{k\eta} \ , \tag{66}$$

where η represents different fermion species. The fermion operators are defined as:

$$c_{k\eta} \left|0\right\rangle \equiv 0, \forall k > 0 \quad c^\dagger_{k\eta} \left|0\right\rangle \equiv 0, \forall k < 0 \ . \tag{67}$$

Then the N-particle ground state can be defined from vacuum sate and fermion operators:

$$\left|\bar{N}\right\rangle_0 \equiv C_1^{N_1} \cdots C_M^{N_M} \left|0\right\rangle \tag{68}$$

$$C_\eta^{N_\eta} \equiv \begin{cases} c^\dagger_{N_\eta \eta} \cdots c^\dagger_{1\eta} & N_\eta > 0 \\ 1 & N_\eta = 0 \\ c_{(N_\eta+1)\eta} \cdots c_{0\eta} & N_\eta < 0 \end{cases} \tag{69}$$

Now define the boson operators as:

$$b^\dagger_{q\eta} \equiv \frac{i}{\sqrt{n_q}} \sum_{k=-\infty}^{\infty} c^\dagger_{k+q\,\eta} c_{k\eta} \quad b_{q\eta} \equiv \frac{-i}{\sqrt{n_q}} \sum_{k=-\infty}^{\infty} c^\dagger_{k-q\,\eta} c_{k\eta} \ . \tag{70}$$

The boson operator shifts the momentum of all the particles in N-particle ground state by q or $-q$. Acting b on the N-particle ground state yields 0, so that $\left|\bar{N}\right\rangle_0$ can be treated as a vacuum state for boson operators. Moreover, boson operators satisfy the expected commutation relation:

$$[b_{q\eta}, b_{q'\eta'}] = [b^\dagger_{q\eta}, b^\dagger_{q'\eta'}] = 0, \quad [N_{q\eta}, b_{q'\eta'}] = [N_{q\eta}, b^\dagger_{q'\eta'}] = 0 \ , \tag{71}$$

where the number operator is defined as $\hat{N}_\eta \equiv \sum_{k=-\infty}^{\infty} : c^\dagger_{k\eta} c_{k\eta} :$. The normal ordering is introduced to make the number operator counting the fermions lying above the fermion sea. The last relation tells that the boson operator does not change the number of fermions in the system, hence it is neccessary to introduce one operator that is able to change the fermion numbers. It is called Kelin factor and defined as:

$$[b_{q\eta}, F^\dagger_{\eta'}] = [b^\dagger_{q\eta}, F^\dagger_{\eta'}] = [b_{q\eta}, F_{\eta'}] = [b^\dagger_{q\eta}, F_{\eta'}] = 0 \tag{72}$$

$$F^\dagger_\eta \left|\mathbf{N}\right\rangle \equiv f(b^\dagger) c^\dagger_{N_{\eta+1}} \left|N_1, \ldots, N_\eta, \ldots, N_M\right\rangle_0 \equiv f(b^\dagger) \hat{T}_\eta \left|N_1, \ldots, N_{\eta+1}, \ldots, N_M\right\rangle_0 \tag{73}$$

$$F_\eta \left|\mathbf{N}\right\rangle \equiv f(b^\dagger) c_{N_{\eta+1}} \left|N_1, \ldots, N_\eta, \ldots, N_M\right\rangle_0 \equiv f(b^\dagger) \hat{T}_\eta \left|N_1, \ldots, N_{\eta-1}, \ldots, N_M\right\rangle_0 \ . \tag{74}$$

The effect of $F^\dagger_\eta$ is clear: it creates one more η-fermion on the lowest possible level of the ground state. From the definition above, we can deduce that the Klein factors are unitary operator which are denoted by U in the literatures. With all the necessary operators in hand, the boson field can be defined as:

$$\varphi_\eta(x) \equiv -\sum_{q>0} \frac{1}{\sqrt{n_q}} e^{-iqx} b_{q\eta} e^{-aq/2} \quad \varphi_\eta^\dagger(x) \equiv -\sum_{q>0} \frac{1}{\sqrt{n_q}} e^{iqx} b_{q\eta}^\dagger e^{-aq/2} \tag{75}$$

$$\phi_\eta(x) \equiv \phi_\eta + \phi_\eta^\dagger \ . \tag{76}$$

The boson fields can be checked (appendix A.5) that they satisfy the appropriate commutation relations:

$$[\varphi_\eta(x), \varphi_{\eta'}(x')] = [\varphi_\eta^\dagger(x), \varphi_{\eta'}^\dagger(x')] = 0 \tag{77}$$

$$[\varphi_\eta(x), \varphi_{\eta'}^\dagger(x')] = -\delta_{\eta\eta'} \ln [i \frac{2\pi}{L}(x - x' - ia)], L \to \infty \ . \tag{78}$$

The fermion density operator can now be expressed in terms of boson fields:

$$\rho_\eta(x) \equiv : \psi_\eta^\dagger(x)\psi_\eta(x) := \partial_x \phi_\eta(x) + \frac{2\pi}{L}\hat{N}_\eta, \quad a \to 0 \ . \tag{79}$$

This is the same form that Coleman claimed in (5). The explicit bosonization identity can be obtained by acting the fermion field on an arbitrary N-particle state, applying the available identities and we are able to find, for $L \to \infty$:

$$\psi_\eta(x) = F_\eta a^{-1/2} e^{-i\phi_\eta(x)} \ . \tag{80}$$

6. Justification and experiment of Luttinger liquid

In this section we would like to give some handwaving arguments toward the analysis of Luttinger liquid.

6.1. **Why 1D?.** The Luttinger model is special because the dispersion relation is linear around the fermi point . Since we are only interested in the low energy excitation, we can linearize the dispersion relation as $\epsilon = v_f(k - k_f), \forall k \in (-\infty, \infty)$. It is not the case for larger dimensions, as in two dimension the fermi points turn into a continuous circle.

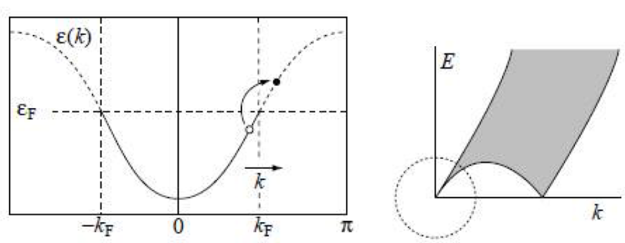

Figure 1. The despersion relation of 1D model can be linearized at fermi point[12].

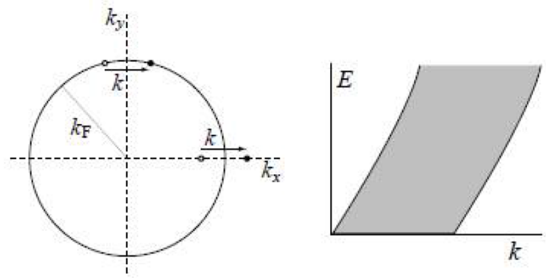

FIGURE 2. In 2D it is not possible to linearize the dispersion relation[12].

In order to visualize a given interacting Hamiltonian, it is better to break the fermion field in Hamiltonian as left and right movers. The fermion field in 1-d wire can be written as:

$$\psi(x) = e^{-ik_F x}\tilde{\psi}_L(x) + e^{ik_F x}\tilde{\psi}_R(x) \ . \tag{81}$$

where the tilde means we extend the range of k from (k_F, ∞) to $(-\infty, \infty)$.

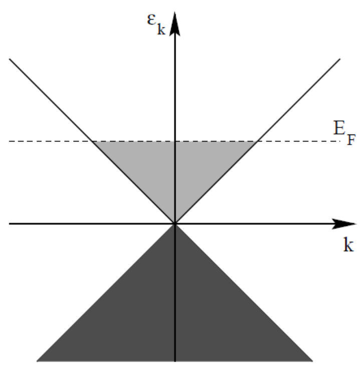

FIGURE 3. single particle spectrum dispersion relation. Occupied states are shown in grey, the dark area shows the unphysical state. From [13].

The state lies way below the fermion sea is unaffected by low energy excitations. The left/right movers have momentum and energy unbounded. This modification makes the 1-D model solvable, in a sense that the Hamiltonian can be written as a free Hamiltonian, even the interactions exist. The electron interaction Hamiltonian is:

$$H = H_{kin} + H_{int} \ . \tag{82}$$

Assuming linear dispersion relation, the kinetic part is (explicit in appendix B.2):

$$H_{kin} = \sum_{\eta=L,R} \sum_{k=-\infty}^{\infty} k : c^{\dagger}_{k\eta} c_{k\eta} := \int_{-L/2}^{L/2} \frac{dx}{2\pi} : \frac{1}{2}[\tilde{\rho}_L^2 + \tilde{\rho}_R^2](x) : \ . \tag{83}$$

For the interaction part (B.2):

$$H_{int} = \frac{1}{2} \int dx dx' \, \psi^{\dagger}(x)\psi^{\dagger}(x')V_{ee}(x-x')\psi(x')\psi(x) \ . \tag{84}$$

Since each ψ has right and left two components, the H_{int} should have 16 terms in total. We drop the 8 terms that do not conserve the number of right and left movers. There are 8 terms left that can be catagorized into two groups. One group is interbranch interaction where the interaction happens between the same fermion species, and the other group is intrabranch interaction where the interaction occurs between different fermion species. We assume the strengh is different for these two kinds of interactions, so eventually we are left with:

$$H_{int} = \int_{-L/2}^{L/2} : g_2 \tilde{\rho}_L(x)\tilde{\rho}_R(x) + \frac{1}{2} g_4[\tilde{\rho}_L^2(x) + \tilde{\rho}_R^2(x)]] : \ . \tag{85}$$

The full Hamiltonian can be expressed as:

$$H_0 = H_{kin} + H_{int} = \frac{v}{4} \int_{-L/2}^{L/2} \frac{dx}{2\pi} : [\frac{1}{g}(\tilde{\rho}_L + \tilde{\rho}_R)^2 + g(\tilde{\rho}_L - \tilde{\rho}_R)^2](x) : \tag{86}$$

$$v \equiv \sqrt{(1+g_4)^2 - g_2^2} \quad g \equiv \sqrt{\frac{1+g_4-g_2}{1+g_4+g_2}} \ .$$

6.2. **What is bizarre?** There are some strange behaviors that belong only to 1D electron gas. Tunneling is a quantity that can be measured experimentally for 1D quantum wires. It occurs between the wire and the tip of a scanning tunneling microscopy. Another example is a 1D wire connected weakly to two leads at ends. One interesting property that determines

FIGURE 4. two tunneling examples. From [14]

the tunneling current is the local density of state, at zero temerature reads:

$$\rho_{dos}(\omega) \equiv \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{i\omega t} \langle G| \, \Psi(t, x=0)\Psi^{\dagger}(0, x=0) \, |G\rangle \ . \tag{87}$$

With the diagonalized free Hamiltonian and boson field Φ_ν ready from last section, the correlation function can be separated as product of two independent correlation functions.

Each one relates to one of the boson species ϕ_ν. the correlation function is found to be proportional to (appendix B.3):

$$\langle G| \Psi(t, x=0)\Psi^\dagger(0, x=0) |G\rangle \sim t^{-\frac{1}{2}(\frac{1}{g}+g)} \tag{88}$$

Thus the local density of state is proportional to:

$$\rho_{dos} \sim \omega^{\nu-1}, \quad \nu = \frac{1}{2}(1/g+g) \ .$$

For $g \neq 1$, $\nu > 1$, so that the tunneling density vanishes at low frequencies. The problem becomes more complex if impurity is added to the site $x = 0$, as it leads to forward and backward scattering. In the presence of impurity, the Hamiltonian has to include two more terms $H_F + F_B$. H_F is for forward scattering, and H_B for backward scattering. The full Hamiltonian can then be separated into two $H_- + H_F$ and $H_+ + H_B$, which can be diagonalized using refermionization. After the refermionization, the full Hamiltonian is the sum of two independent free Hamiltonians. Up to this point, the tunneling density can be treated the same way as this section does for non impurity situtation. It is found [9] that for $g = 1/2$, the correlation function is proportional to t^{-2}, so that $\rho(\omega) \sim \omega$.

The tunneling density of states of Luttinger liquid would lead to an interesting phenomenon that is known as orthogonality catastrophe, or power-law suppression of the tunneling differential conductance. The differential tunneling conductance $G(V, T)$ is proportional to ρ_{dos}. Thus for $T = 0$, $G \sim \omega^{\nu-1} \sim V^{\nu-1}$. For $\nu \neq 1$, we must have vanishing conductance at low bias. When the bias becomes larger compared with the temperature, the normal Ohmic behavior of condutance appears again.

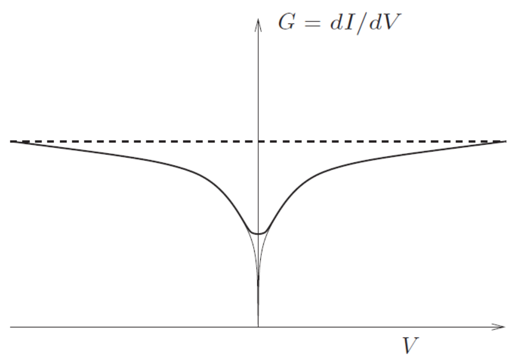

FIGURE 5. The differential tunneling conductance as a function of bias. From [8]

Its momentum distribution function has a jump at the fermi surface. Unlike the normal fermi liquid where the distribution function vanishes completely at fermi surface. The width of the jump is related to the weight of quasiparticle Z, in this case $Z < 1$. The stronger the interaction, the smaller the weight of quasiparticle in excitation. For Luttinger liquid, $Z \sim 0$ indicates there is actually no quasiparticle excitations. The excitation takes the form of particle hole pair that are strongly correlated.

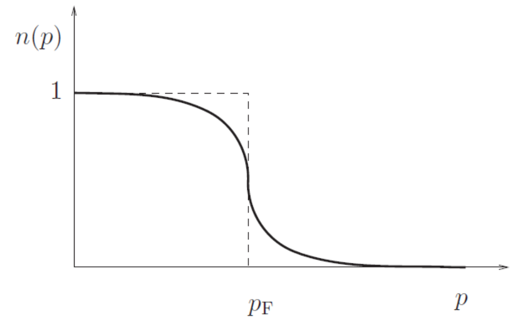

FIGURE 6. The momentum distribution function of Luttinger liquid. There is a small discontinuity at the fermi surface. From [8]

For Luttinger liquid invovles spin, the Hamiltonian becomes more complicated to bosonize and diagonalize. However, we can give a short argument for spin-charge separation in terms of Hamiltonian represented in momentum space:

$$H_{kin} \sim \sum_{\sigma,q} [\rho_{R\sigma}(q)\rho_{R\sigma}(-q) + \rho_{L\sigma}(q)\rho_{L\sigma}(-q)] \tag{89}$$

$$H_{int} \sim \sum_{q,\sigma,\sigma'} [\rho_{R\sigma}(q) + \rho_{L\sigma}(q)][\rho_{R\sigma'}(-q) + \rho_{L\sigma'}(-q)]\ . \tag{90}$$

Now if we define the spin and charge densities as

$$\rho = \rho_\uparrow + \rho_\downarrow \quad \sigma = \rho_\uparrow - \rho_\downarrow\ . \tag{91}$$

Then the Hamiltonian becomes:

(92)

$$H_{kin} \sim \sum_q [\rho_R(-q)\rho_R(q) + \rho_L(q)\rho_L(-q) + \sigma_R(-q)\sigma_R(q) + \sigma_L(q)\sigma_R(q) + \sigma_L(q)\sigma_L(-q)]$$

(93)

$$H_{int} \sim \sum_q [\rho_R(q) + \rho_L(q)][\rho_R(-q) + \rho_L(-q)]\ .$$

The charge density and spin density operators have decoupled from each other in the kinetic term, and the interaction Hamiltonian only contains charge density terms. If we continue to bosonize this Hamiltonian with new boson fields, it is not suprising that we can get two decoupled Hamiltonian H_σ and H_ρ such that charge and spin oscillations propagate with different velocities. This is the spin-charge separation, this refers that two excitations are moving independently of one another.

To visualize the spin-charge separation, consider the cartoon:

After the system undergoes a particle-hole excitation, the point labeled by holon has a net charge but zero net spin, and the point labeled by spinon has a net spin but zero net charge. The holon and spinon are independent of each other, and their motion are not

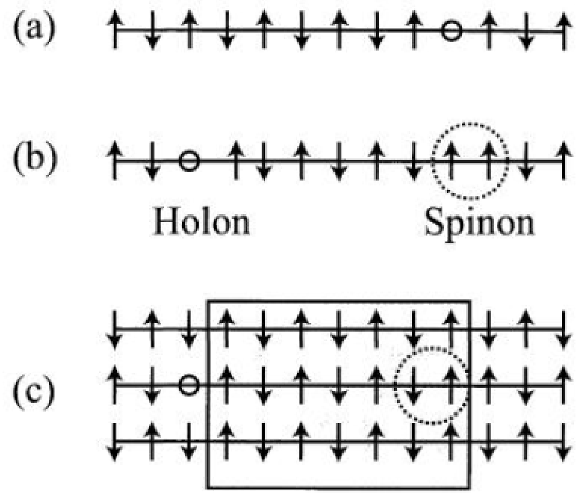

FIGURE 7. (a) has a particle-hole excitation (b) spin and charge are separated (c) in higher dimension such phenomenon breaks down [15]

constrained to each other. In higher dimension, in order to create a spinon, we have to align electron with neighbors of the same spin.

7. Conclusion

In this note we reviewed the development of bosonization technique both in particle and solid state physics. However, there are more can be reviewed under this catalogue: the non abelian bosonization, the scaling of operator product expansion, the universality classes of 1D electron liquid, the non-linear spectrum Luttinger model, the application of Luttinger liquid to 1D Hubbard model, the Kondo model, and possible comformal field theory. Moreover, bosonization is only one of the many famous methods that solve the 1D system. It might be true that we have only seen an iceberg of the one dimensional world, and this is a branch of physics in which both theorists and experimentalists could found amazing physics that is different from what we usully see in higher dimensions.

References

[1] S. Coleman, Phy. Rev. D 11 (1975) 2088
[2] S. Mandelstam, phys. rev. D11 (1975) 3026
[3] E. D. Witten, Commun. Math. Phys. 92 (1984) 455
[4] A. Belavin, A. Polyakov, A. Zamolodchikov, Nuclear Phys. B, 241 (2) (1984)
[5] S. Tomonaga, Prog. Theor. Phys. (Kyoto) 5 (1950) 544
[6] P. Jordan and E. Wigner, Z. Physik 47, 631 (1928)
[7] A. Luther and I. Peschel, Phys. Rev. B 12, 3908 (1975)
[8] E. Frandkin, *"Field Theories of Condensed Matter Physics "*, Cambridge University Press, 2013

[9] Jan Von Delft, Herbert Schoeller, *Bosonization for Beginners - Refermionization for Experts.* cond-mat/9805275v3
[10] P. D. Francesco, *"Conformal Field Theory"*, Springer, 1996
[11] M. Srednicki, *"An Introduction to Quantum Field Theory"*, Cambridge University Press, 2012
[12] D. Senechal, *An introduction to bosonization.* cond-mat/9908262v1
[13] H.J. Schulz, G. Cuniberti, P. Pieri, *Fermi liquids and Luttinger liquids.* cond-mat/9807366v2
[14] S. Eggert, *One-dimensional quantum wires: A pedestrian approach to bosonization.* arXiv:0708.0003v3
[15] T. Giamarchi, *Quantum Physics in One Dimension.* Clarendon Press, 2003

Appendix A. Bosonization identity in condensed matter

In this appendix we followed Delft's [9] constructive approach to construct the bosonization identity. For new comers this approach is preferred since the bozonization identity, $\psi_\eta(x) \sim F_\eta e^{-i\phi_\eta(x)}$, is better understood from its origin. The beginning point is to set a fermion field $\psi_\eta(x) = (\frac{2\pi}{L})^{1/2}\sum_k e^{-ikx}c_{k\eta}$ within a system of length L. Then all other operators can be constructed in terms of $c_{k\eta}$ and have their physical interpretation.

A.1. **Fermion operator and field.** Let the fermion creation and annihilation operators to have the canonical anticommutation relations:

$$\{c_{k\eta}, c^\dagger_{k'\eta'}\} = \delta_{\eta\eta'}\delta_{kk'}, \quad k \in [-\infty, \infty], \quad \eta = 1, \ldots, M \tag{94}$$

η is a species index, it can denote spin or left/right-moving electrons. The momntum index k is

$$k = \frac{2\pi}{L}(n_k - \frac{1}{2}\delta_b), \quad n_k \in \mathcal{Z}, \quad \delta_b \in [0, 2) \tag{95}$$

δ_b is useful in determining the boundary conditions of the fermion fields, where the fermion field is defined as:

$$\psi_\eta(x) \equiv (\frac{2\pi}{L})^{1/2}\sum_{k=-\infty}^{\infty} e^{-ikx}c_{k\eta} \tag{96}$$

$$\Rightarrow c_{k\eta} = (2\pi L)^{-1/2}\int_{-L/2}^{L/2} dx\, e^{ikx}\psi_\eta(x) \tag{97}$$

From (95) and (96) it is easy to see:

$$\psi_\eta(x + L/2) = e^{i\pi\delta_b}\psi_\eta(x - L/2) \tag{98}$$

Thus $\delta_b = 1(0)$ it gives anti-periodic(periodic) condition. To confirm the anticommutation relation of the constructed fermion fields, using (94) and (95) we have:

$$\begin{aligned}\{\psi_\eta(x), \psi^\dagger_{\eta'}(x')\} &= \delta_{\eta\eta'}\frac{2\pi}{L}\sum_{n\in\mathcal{Z}} e^{-i\frac{2\pi}{L}(n_k - \frac{1}{2}\delta_b)(x-x')} \\ &= \delta_{\eta\eta'}e^{i\frac{\pi}{L}\delta_b(x-x')}\delta((x'-x) - nL)\end{aligned} \tag{99}$$

In the second line we used the identity $\sum_{n\in\mathcal{Z}} e^{iny} = 2\pi\sum_{n\in\mathcal{Z}}\delta(y - 2\pi n)$. Thus we see that the expected anticommutation relation is satisfied.

A.2. **Vacuum and N-particle ground state.** The vacuum state $|0\rangle$ is defined to be:

$$c_{k\eta}|0\rangle \equiv 0, \forall k > 0 \quad c^\dagger_{k\eta}|0\rangle \equiv 0, \forall k < 0 \tag{100}$$

It says at vacuum state no electron or positron can be annihilated. The vacuum state is the fermi sea, and is used as a reference state. The fermion-normal-ordering is defined with respect to the fermi sea:

$$: ABC \cdots := ABC - \langle 0| ABC \ldots |0\rangle \,, \quad A, B, C \cdots \in \{c_{k\eta}, c^\dagger_{k\eta}\} \tag{101}$$

The fermion-normal-ordering is not the same as the timer-normal-ordering. It moves fermion creation(annihilation) operator with negative(positive) momentum to the right of other operators. To give it a check, let $k > 0 \wedge k' < 0$,

$$: c^\dagger_k c_{k'} := -c_{k'} c^\dagger_k = c^\dagger_k c_{k'} - \delta_{kk'} = c^\dagger_k c_{k'} - \langle c^\dagger_k c_{k'} \rangle_0$$

Let $\hat{N}_\eta \equiv \sum_{k=-\infty}^{\infty} : c^\dagger_{k\eta} c_{k\eta} := \sum_{k=-\infty}^{\infty} [c^\dagger_{k\eta} c_{k\eta} - \langle c^\dagger_{k\eta} c_{k\eta} \rangle_0]$ be the number operator that counts the number of η electron with respect to the fermion sea. The eigenvalue of $\hat{N}_\eta$ running through all η should be a vector $\mathbf{N} = (N_1, \cdots, N_M) \in \mathcal{Z}^M$. This makes sense since the eigenvalue of $c^\dagger_{k\eta} c_{k\eta}$ is either 1 or 0, and by subtstracting $\langle c^\dagger_{k\eta} c_{k\eta} \rangle_0$ we only count the number of electrons above the fermion sea or positrons under the fermion sea. Denote $|\mathbf{N}\rangle_0$ as the N-particle ground state relative to the fermion sea, and no particle-hole excitation is allowed:

$$|\bar{N}\rangle_0 \equiv C_1^{N_1} \cdots C_M^{N_M} \, |0\rangle \tag{102}$$

$$C_\eta^{N_\eta} \equiv \begin{cases} c^\dagger_{N_\eta \eta} \cdots c^\dagger_{1\eta} & N_\eta > 0 \\ 1 & N_\eta = 0 \\ c_{(N_\eta+1)\eta} \cdots c_{0\eta} & N_\eta < 0 \end{cases} \tag{103}$$

Based on our construction, all N-particle excited state can be built from N-particle ground state $|\mathbf{N}\rangle_0$, since they have the same number of particles for each species.

A.3. Bosonic operator.

Define the bosonc creation and annihilation operators as:

$$b^\dagger_{q\eta} \equiv \frac{i}{\sqrt{n_q}} \sum_{k=-\infty}^{\infty} c^\dagger_{k+q\,\eta} c_{k\eta} \quad b_{q\eta} \equiv \frac{-i}{\sqrt{n_q}} \sum_{k=-\infty}^{\infty} c^\dagger_{k-q\,\eta} c_{k\eta} \tag{104}$$

where $q \equiv \frac{2\pi}{L} n_q, n_q \in \mathcal{Z}^+$, so that q is positive. It can be seen that $b^\dagger_{q\eta}$ shift all the η electrons' momentum by q, if it acts on $|\mathbf{N}\rangle$. Similarly for $b_{q\eta}$ which lowers the momentum by q. The normalization constant gives the expected commutation relation (suppressing the η indices):

$$[b_{q\eta}, b^\dagger_{q'\eta'}] = \frac{1}{n_q} \sum_k [c^\dagger_{k-q} c_{k-q'} - c^\dagger_{k-q+q'} c_k] \tag{105}$$

If $q \neq q'$, it is normal-ordered because the vacuum state does not come into play. We can let $k + q' \to k$ so that it yields 0. If $q = q'$, we have to normal-roder it first:

$$[b_{q\eta}, b^\dagger_{q'\eta'}] = \delta_{qq'} \sum_k \frac{1}{n_q} \{[: c^\dagger_k c_k : - : c^\dagger_{k+q} c_{k+q} :] + [\langle c^\dagger_k c_k \rangle_0 - \langle c^\dagger_{k+q} c_{k+q} \rangle_0]\} \tag{106}$$

The first term yields 0 because it is normal ordered. The second term, based on our definition of fermion creation/annihilation operator acting on the vacuum state (100), gives:

$$\frac{1}{n_q}\left(\sum_{n_k=-\infty}^{0} - \sum_{n_k=-\infty}^{-n_q}\right) = \frac{1}{n_q}n_q = 1 \tag{107}$$

Therefore, we obtained $[b_{q\eta}, b^\dagger_{q'\eta'}] = \delta_{qq'}\delta_{\eta\eta'}$. Also the remaining relations are easy to check:

$$[b_{q\eta}, b_{q'\eta'}] = [b^\dagger_{q\eta}, b^\dagger_{q'\eta'}] = 0, \quad [N_{q\eta}, b_{q'\eta'}] = [N_{q\eta}, b^\dagger_{q'\eta'}] = 0 \tag{108}$$

Based on (102) and the anticommutation relation of fermion creation/annihilation operators, we can deduce $b_{q\eta}\left|\mathbf{N}\right\rangle_0 = 0$ for all q, η. This makes sense since for ground state there is no excitation, down shifting the momentum of every particle/hole by a positive value q gives 0. Therefore, we can treat $\left|\mathbf{N}\right\rangle_0$ as the vacuum state for bosonic operator as to $\left|0\right\rangle$ for fermion operator, and the normal ordering for bosonic operator can be similarly defined as:

$$: ABC\cdots := ABC\cdots - {}_0\left\langle\mathbf{N}\right| ABC\ldots\left|\mathbf{N}\right\rangle_0 \tag{109}$$

Clearly as $\left|\mathbf{N}\right\rangle_0$ is interchangeable with $\left|0\right\rangle$, the operator product is boson-normal-ordered if and only if it is fermion-normal-ordered. It is obvious that every excited state $\left|\mathbf{N}\right\rangle$ is a linear combination of bilinear fermion operators acting on the ground state:

$$\left|\mathbf{N}\right\rangle = f(c^\dagger_{k\eta}c_{k'\eta})\left|\mathbf{N}\right\rangle_0 \tag{110}$$

Remarkably this statement is also nontrivially true for a linear combination of bosonic operators $b^\dagger_{q\eta}$, as was proved by Haldane [**?**]. Here we take the proof, and a profound consequence can be seen from this statement. If we write the bilinear fermion operator as bilinear fermion field operator, by (97)

$$c^\dagger_{k\eta}c_{k'\eta} = \frac{1}{2\pi L}\int dx \int dx'\, e^{i(k'x'-kx)}\psi^\dagger_\eta(x)\psi_\eta(x') \tag{111}$$

This indicates that $\psi^\dagger_\eta(x)\psi_\eta(x')$ can be written as bosonic operators, and shows the sign of possibility of building a bosonization rule.

A.4. **Klein factors.** From (108) since the bosonic operator commute with the number operator, the bosonic operator does not change the number of fermions of a given state. Hence for the completeness of bosonization, it is necessary to include operators that could change the number of fermions. The Klein factor $F^\dagger_\eta$ and F_η are introduced for this purpose, where the notation comes from Kotliar and Si [**?**]. First it is required that they commute with bosonic operators

$$[b_{q\eta}, F^\dagger_{\eta'}] = [b^\dagger_{q\eta}, F^\dagger_{\eta'}] = [b_{q\eta}, F_{\eta'}] = [b^\dagger_{q\eta}, F_{\eta'}] = 0 \tag{112}$$

Then we require

$$F_\eta^\dagger \left|\mathbf{N}\right\rangle \equiv f(b^\dagger) c^\dagger_{N_{\eta+1}} \left|N_1, \ldots, N_\eta, \ldots, N_M\right\rangle_0 \equiv f(b^\dagger)\hat{T}_\eta \left|N_1, \ldots, N_{\eta+1}, \ldots, N_M\right\rangle_0 \tag{113}$$

$$F_\eta \left|\mathbf{N}\right\rangle \equiv f(b^\dagger) c_{N_{\eta+1}} \left|N_1, \ldots, N_\eta, \ldots, N_M\right\rangle_0 \equiv f(b^\dagger)\hat{T}_\eta \left|N_1, \ldots, N_{\eta-1}, \ldots, N_M\right\rangle_0 \tag{114}$$

It can be seen that $F_\eta^\dagger \left|\mathbf{N}\right\rangle$ has the same excitation as $\left|\mathbf{N}\right\rangle$, because it is the same $f(b^\dagger)$ acting on ground states. They differ by the ground state in which $F_\eta^\dagger \left|\mathbf{N}\right\rangle$ has one more η-electron in the lowest empty level of $\left|\mathbf{N}\right\rangle$. $\hat{T}_\eta = (-1)^{\sum_{i=1}^{\eta-1} \hat{N}_i}$ is a phase factor we need to take into account when $c^\dagger_{N_{\eta+1}}$ passes by all the fermion creation operators in front of $C_\eta^{N_\eta}$. From (113) and (114), $F_\eta^\dagger F_\eta \left|\mathbf{N}\right\rangle = \left|\mathbf{N}\right\rangle$, hence the Klein factor is unitary. Other commutation relations are (easy to check):

$$\{F_\eta, F_\eta^\dagger\} = 2\delta_{\eta\eta'}, \quad \hat{N}_\eta F_\eta^\dagger = F_\eta^\dagger(\hat{N}_\eta + 1) \Rightarrow [\hat{N}_{\eta'}, F_\eta^\dagger] = \delta_{\eta'\eta} F_\eta^\dagger, \quad [\hat{N}_{\eta'}, F_\eta] = -\delta_{\eta'\eta} F_\eta \tag{115}$$

A.5. **Boson field.** With all the operators properly defined for the bosonic ground state $\left|\mathbf{N}\right\rangle$, we are now able to give definition to the boson field:

$$\varphi_\eta(x) \equiv -\sum_{q>0} \frac{1}{\sqrt{n_q}} e^{-iqx} b_{q\eta} e^{-aq/2} \quad \varphi_\eta^\dagger(x) \equiv -\sum_{q>0} \frac{1}{\sqrt{n_q}} e^{iqx} b^\dagger_{q\eta} e^{-aq/2} \tag{116}$$

$$\phi_\eta(x) \equiv -\sum_{q>0} [\frac{1}{\sqrt{n_q}} e^{-iqx} b_{q\eta} + \frac{1}{\sqrt{n_q}} e^{iqx} b^\dagger_{q\eta}] e^{-aq/2} \tag{117}$$

Then the electron density operator can be straightforwardly expressed as boson fields:

$$\rho_\eta(x) \equiv : \psi_\eta^\dagger(x)\psi_\eta(x) := \frac{2\pi}{L} \sum_q e^{-iqx} \sum_k : c^\dagger_{k-q,\eta} c_{k\eta} : \tag{118}$$

$$= \frac{2\pi}{L} \sum_{q>0} i\sqrt{n_q}(e^{-iqx} b_{q\eta} - e^{iqx} b^\dagger_{q\eta}) + \frac{2\pi}{L} \sum_k : c^\dagger_{k,\eta} c_{k\eta} : \tag{119}$$

$$= \partial_x \phi_\eta(x) + \frac{2\pi}{L} \hat{N}_\eta, \quad a \to 0 \tag{120}$$

The commutation relations of boson fields can be checked:

$$[\varphi_\eta(x), \varphi_{\eta'}(x')] = [\varphi_\eta^\dagger(x), \varphi^\dagger_{\eta'}(x')] = 0 \tag{121}$$

$$[\varphi_\eta(x), \varphi^\dagger_{\eta'}(x')] = \delta_{\eta\eta'} \sum_{q>0} \frac{1}{n_q} e^{-q[i(x-x')+a]} = -\delta_{\eta\eta'} \ln[1 - e^{-i\frac{2\pi}{L}(x-x'-ia)}] = -\delta_{\eta\eta'} \ln[i\frac{2\pi}{L}(x-x'-ia)], L \to \infty \tag{122}$$

Notice the importance of a in (122) when $x = x'$, where a is treated as a UV cut off to regularize the result. With (122) and Weyl's identity, we can easily show:

$$e^{i\varphi_\eta^\dagger(x)} e^{i\varphi_\eta(x)} = (\frac{L}{2\pi a})^{1/2} e^{i\phi_\eta(x)} \quad e^{-i\varphi_\eta(x)} e^{-i\varphi_\eta^\dagger(x)} = (\frac{L}{2\pi a})^{1/2} e^{-i\phi_\eta(x)} \tag{123}$$

Using (117) and (122), the commutator of $\phi_\eta(x)$ with its derivative can be find (to order of $\frac{1}{L}$), as taking the limit $L \to \infty$ first and $a \to 0$ second:

$$[\phi_\eta(x), \partial_{x'}\phi_{\eta'}(x')] = \delta_{\eta\eta'} 2\pi i[\delta(x - x') - \frac{1}{L}] \tag{124}$$

A.6. **Bosonization identity.** We are now able to find the bosonization identity by applying the fermion field on both boson vacuum state and arbitrary boson states. Let $\alpha_q(x) \equiv \frac{i}{\sqrt{n_q}} e^{iqx}$, and write both $\psi_\eta(x)$ and $b_{q\eta}$ in terms of fermion operators. The commutation relation can be seen as:

$$[b_{q\eta'}, \psi_\eta(x)] = \delta_{\eta\eta'}\alpha_q(x)\psi_\eta(x) \quad [b^\dagger_{q\eta'}, \psi_\eta(x)] = \delta_{\eta\eta'}\alpha^*_q(x)\psi_\eta(x) \tag{125}$$

Since it yields 0 for boson annihilation operator acting on the boson vacuum state, with (125) we immediately have:

$$b_{q\eta'}\psi_\eta(x) \left|\mathbf{N}\right\rangle_0 = \delta_{\eta\eta'}\alpha_q(x)\psi_\eta(x) \left|\mathbf{N}\right\rangle_0 \tag{126}$$

Equation (126) tells that $\psi_\eta(x) \left|\mathbf{N}\right\rangle_0$ is an eigenstate of $b_{q\eta'}$, therefore $\psi_\eta(x) \left|\mathbf{N}\right\rangle_0$ must can be expressed in the form:

$$\psi_\eta(x) \left|\mathbf{N}\right\rangle_0 = e^{\sum_{q>0} \alpha_q(x) b^\dagger_{q\eta}} F_\eta \hat{\lambda}_\eta(x) \left|\mathbf{N}\right\rangle_0 = e^{-i\varphi^\dagger_\eta(x)} F_\eta \hat{\lambda}_\eta(x) \left|\mathbf{N}\right\rangle_0 \tag{127}$$

The validity of (127) can be checked by inserting it into (126), with the identity $C = [A, B] \Rightarrow [A, e^B] = Ce^B, A = b_{q\eta'}, B = -i\phi^\dagger_\eta(x), C = \delta_{\eta\eta'}\alpha_q(x)$. The value of $\hat{\lambda}_\eta(x)$ can be determined as follow, as noticing first that ${}_0\left\langle\mathbf{N}\right| F^\dagger_\eta \psi_\eta(x) \left|\mathbf{N}\right\rangle_0 = \lambda_\eta(x)$. Then we can easily find, by writing $\psi_\eta(x)$ as combination of fermion operators, that

$${}_0\left\langle\mathbf{N}\right| F^\dagger_\eta \psi_\eta(x) \left|\mathbf{N}\right\rangle_0 = (\frac{2\pi}{L})^{1/2} e^{-i\frac{2\pi}{L}(N_\eta - \frac{1}{2}\delta_b)x} \tag{128}$$

Two more identities are needed, let $\left|\mathbf{N}\right\rangle = f(\{b^\dagger_{q\eta'}\}) \left|\mathbf{N}\right\rangle_0$, then

$$f(\{b^\dagger_{q\eta'}\}) \left|\mathbf{N}\right\rangle_0 = f(\{b^\dagger_{q\eta'} - \delta_{\eta\eta'}\alpha^*_q(x)\})\psi_\eta(x) \tag{129}$$

$$f(\{b^\dagger_{q\eta'} - \delta_{\eta\eta'}\alpha^*_q(x)\}) = e^{-i\varphi_\eta(x)} f(\{b^\dagger_{q\eta'}\}) e^{i\varphi_\eta(x)} \tag{130}$$

Equation (129) uses the identity $[A, B] = DB \wedge [A, D] = [B, D] = 0 \Rightarrow f(A)B = Bf(A+D)$, and (130) uses the identity $c = [A, B] \wedge [A, C] = [B, C] = 0 \Rightarrow e^{-B} f(A) e^B = f(A+C)$.

The final step is to evaluate $\psi_\eta(x)\,|\mathbf{N}\rangle$:

(131)
$$\psi_\eta(x)\,|\mathbf{N}\rangle = \psi_\eta(x) f(\{b^\dagger_{q\eta'}\})\,|\mathbf{N}\rangle_0 = f(\{b^\dagger_{q\eta'} - \delta_{\eta\eta'}\alpha^*_q(x)\})\psi_\eta(x)\,|\mathbf{N}\rangle_0 \quad \text{used (129)}$$

$$= f(\{b^\dagger_{q\eta'} - \delta_{\eta\eta'}\alpha^*_q(x)\}) e^{-i\varphi^\dagger_\eta(x)} F_\eta \hat{\lambda}_\eta(x)\,|\mathbf{N}\rangle_0 \quad \text{used the relation (127) } \psi_\eta\,|\mathbf{N}\rangle_0 \tag{132}$$

$$= F_\eta \hat{\lambda}_\eta(x) e^{-i\varphi^\dagger_\eta(x)} f(\{b^\dagger_{q\eta'} - \delta_{\eta\eta'}\alpha^*_q(x)\})\,|\mathbf{N}\rangle_0 \quad \text{since } F_\eta \text{ commutes with } b^\dagger_{q\eta'} \tag{133}$$

$$= F_\eta \hat{\lambda}_\eta(x) e^{-i\varphi^\dagger_\eta(x)} [e^{-i\varphi_\eta(x)} f(\{b^\dagger_{q\eta'}\}) e^{i\varphi_\eta(x)}]\,|\mathbf{N}\rangle_0 \quad \text{used (130)} \tag{134}$$

$$= F_\eta \hat{\lambda}_\eta(x) e^{-i\varphi^\dagger_\eta(x)} e^{-i\varphi_\eta(x)} f(\{b^\dagger_{q\eta'}\})\,|\mathbf{N}\rangle_0 \quad \text{since } b_{q\eta'}\,|\mathbf{N}\rangle_0 = 0 \tag{135}$$

$$= F_\eta \hat{\lambda}_\eta(x) e^{-i\varphi^\dagger_\eta(x)} e^{-i\varphi_\eta(x)}\,|\mathbf{N}\rangle \tag{136}$$

Since the choice of $|\mathbf{N}\rangle$ is arbitrary, we conclude that

$$\psi_\eta(x) = F_\eta \hat{\lambda}_\eta(x) e^{-i\varphi^\dagger_\eta(x)} e^{-i\varphi_\eta(x)} \tag{137}$$

$$= F_\eta (\frac{2\pi}{L})^{1/2} e^{-i\frac{2\pi}{L}(N_\eta - \frac{1}{2}\delta_b)x} e^{-i\varphi^\dagger_\eta(x)} e^{-i\varphi_\eta(x)} \tag{138}$$

$$= F_\eta (\frac{2\pi}{L})^{1/2} e^{-i\frac{2\pi}{L}(N_\eta - \frac{1}{2}\delta_b)x} e^{-i\phi_\eta(x)} \quad \text{used (123)} \tag{139}$$

This is the final result of bosonization identity, and cearly for $L \to \infty$, we have a better looking result $F_\eta(\frac{2\pi}{L})^{1/2} e^{-i\phi_\eta(x)}$.

Appendix B. Luttinger model (heuristic view)

In this section we go through the tunneling density of states at in a Luttinger liquid, the discussion will not be explicit due to its length and the time constraint I have.

B.1. **Fermion and boson left and right movers.** The physical fermion field in a 1-D wire of length L can be defined as:

$$\psi_{phy}(x) \equiv (\frac{2\pi}{L})^{1/2} \sum_{p=-\infty}^{\infty} e^{ipx} c_p \tag{140}$$

If we let $p = \mp(k + k_F)$, the above field can be written as:

$$\psi_{phy}(x) = (\frac{2\pi}{L})^{1/2} \sum_{k=-k_F}^{\infty} (e^{-i(k_F+k)x} c_{-(k_F+k)} + e^{i(k_F+k)x} c_{k_F+k}) \tag{141}$$

$$\equiv (\frac{2\pi}{L})^{1/2} \sum_{k=-k_F}^{\infty} (e^{-i(k_F+k)x} c_{k,L} + e^{i(k_F+k)x} c_{k,R}) \tag{142}$$

$$\to (\frac{2\pi}{L})^{1/2} \sum_{k=-\infty}^{\infty} (e^{-i(k_F+k)x} c_{k,L} + e^{i(k_F+k)x} c_{k,R}) \tag{143}$$

$$\equiv e^{-ik_F x} \tilde{\psi}_L(x) + e^{ik_F x} \tilde{\psi}_R(x) \tag{144}$$

In the last line we extended the range of k from $[k_F, \infty)$ to $(-\infty, \infty)$. The inclusion of unphysical state is possible, they are unaffected by the low energy excitation. The tilde means it is a mathematical left/right mover, and they shall only be used in low energy situations. Clearly we have to quantize the momentum k by imposing boundary conditions, we can take $\delta_b = 1$ in (95) to have antiperiodic condition.

Having the fermion left/right operators properly defined, we can follow exactly what we did in appendix A.3 to define the bosonic left/right operators $b_{q\,L/R}$ in terms of $c_{k\,L/R}$. Then the Klein operator, number operator could all be written in terms of left/right movers. Here we have:

$$\tilde{\phi}_{L/R}(x) \equiv -\sum_{n_q>0} \frac{1}{\sqrt{n_q}} e^{-aq/2} [e^{\mp iqx} b_{q\,L/R} + e^{\pm iqx} b^\dagger_{q\,L/R}] \quad \text{from (117), } q = \tfrac{2\pi}{L} n_q \tag{145}$$

$$\tilde{\psi}_{L/R}(x) = a^{-1/2} F_{L/R} e^{\mp i \frac{2\pi}{L}(\hat{N}_{L/R} - \frac{1}{2}\delta_b)x} e^{-i\tilde{\phi}_{L/R}(x)} \quad \text{compare with (123)} \tag{146}$$

$$\tilde{\rho}_{L/R}(x) \equiv : \tilde{\psi}^\dagger_{L/R}(x) \tilde{\psi}_{L/R}(x) := \pm \partial_x \tilde{\phi}_{L/R}(x) + \frac{2\pi}{L} \hat{N}_{L/R} \quad \text{compare with (120)} \tag{147}$$

Notice the plus minus sign in the above equations, the reason is that the left and right movers in (144) are differed by a sign of x in their phase factors.

B.2. **Diagonalizing the electron interaction Hamiltonian by bosonization.** One of the important usages of bosonization is to diagnonalize the interaction Hamiltonian and result in a free Hamiltonian. Here we start with a local electron-electron interaction, bosonize it and in the end make use of a Bogoliubov transformation to get a free Hamiltonian.

Assuming linear dispersion relation, the kinetic term of the Hamiltonian is:

$$H_{kin} = \sum_{\eta=L,R} \sum_{k=-\infty}^{\infty} k : c^\dagger_{k\eta} c_{k\eta} := \int_{-L/2}^{L/2} : [\tilde{\psi}^\dagger_L(x)(i\partial_x)\tilde{\psi}_L(x) + \tilde{\psi}^\dagger_R(x)(-i\partial_x)\tilde{\psi}_R(x)] : \tag{148}$$

$$= \sum_{\nu=L,R} [\frac{2\pi}{L} \frac{1}{2} \hat{N}_\nu^2 + \int_{-L/2}^{L/2} \frac{dx}{2\pi} : \frac{1}{2} (\partial_x \phi_\nu(x))^2 :] = \int_{-L/2}^{L/2} \frac{dx}{2\pi} : \frac{1}{2} [\tilde{\rho}_L^2 + \tilde{\rho}_R^2](x) : \tag{149}$$

In (149) there are some subtleties for the first equality and we do not discuss here, and the second equality comes from (120). The interaction term can be expressed as:

(150)
$$H_{int} = \frac{1}{2L}\sum_{kk'q} V_q c^\dagger_{k-q} c^\dagger_{k'+q} c_{k'} c_k$$

(151)
$$= \frac{1}{2}\int dxdx'\, \psi^\dagger(x)\psi^\dagger(x')V_{ee}(x-x')\psi(x')\psi(x), \quad \text{let } V_{ee}(x-x') = \delta(x-x')$$

(152)
$$\to \frac{1}{2}\int dxdx'\, \psi^\dagger_{phy}(x)\psi^\dagger_{phy}(x')V_{ee}(x-x')\psi_{phy}(x')\psi_{phy}(x), \quad \text{substitute (144) into it}$$

(153)

$$\begin{aligned} = \frac{1}{2}\int \frac{dx}{2\pi}\frac{dx'}{2\pi}\delta(x-x')\{ & \frac{g_4}{2}[\psi^\dagger_R(x)\psi^\dagger_R(x')\psi_R(x')\psi_R(x) + R \leftrightarrow L] \\ & + g_2[\psi^\dagger_R(x)\psi^\dagger_L(x')\psi_L(x')\psi_R(x) + R \leftrightarrow L \\ & + e^{-2ik_F x}[\psi^\dagger_R(x)\psi_R(x')\psi^\dagger_R(x')\psi_L(x) + R \leftrightarrow L] \\ & + e^{-4ik_F x}[\psi^\dagger_R(x)\psi^\dagger_R(x')\psi_L(x')\psi_L(x) + R \leftrightarrow L]\} \end{aligned} \tag{154}$$

Only the first two terms conserve the number of right and left movers, and the last two terms can be dropped since their phase factors are in 'high-energy' region. Thus we have:

$$H_{int} = \int_{-L/2}^{L/2} : g_2\tilde{\rho}_L(x)\tilde{\rho}_R(x) + \frac{1}{2}g_4[\tilde{\rho}^2_L(x) + \tilde{\rho}^2_R(x)]] : \tag{155}$$

The Hamiltonian then is:

$$H_0 = H_{kin} + H_{int} = \frac{v}{4}\int_{-L/2}^{L/2}\frac{dx}{2\pi} : [\frac{1}{g}(\tilde{\rho}_L + \tilde{\rho}_R)^2 + g(\tilde{\rho}_L - \tilde{\rho}_R)^2](x) : \tag{156}$$
$$v \equiv \sqrt{(1+g_4)^2 - g_2^2} \quad g \equiv \sqrt{\frac{1+g_4-g_2}{1+g_4+g_2}}$$

Substitute ρ in terms of boson operators using (147):

(157)
$$H_0 = \frac{2\pi}{L}\frac{v}{2}\{(\frac{1}{g}+g)\sum_{\nu=L,R}[\frac{1}{2}\hat{N}_\nu^2 + \sum_q n_q b^\dagger_{q\nu} b_{q\nu}] + (\frac{1}{g}-g)[\hat{N}_L\hat{N}_R - \sum_q n_q(b_{qR}b_{qL} + b^\dagger_{qR}b^\dagger_{qL})]\}$$

(158)
$$= v\frac{2\pi}{L}\sum_{\nu=\pm}[g^\nu \hat{N}_\nu^2 + \sum_q n_q B^\dagger_{q\nu}B_{q\nu}] \quad \text{by Bogoliubov transformation}$$

(159)
$$= v\sum_{\nu=\pm}[\frac{2\pi}{L}g^\nu\hat{N}_\nu^2 + \int_{-L/2}^{L/2}\frac{dx}{2\pi} : \frac{1}{2}(\partial_x\Phi_\nu(x))^2 :] \quad \text{reproduce the previous line by subsituting in } \Phi_\nu(x)$$

(160)
$$\equiv H_{0+} + H_{0-}$$

where we have the new transformed boson operators:

$$B_{q\pm} \equiv \frac{1}{\sqrt{8}}\{(\frac{1}{\sqrt{g}}+\sqrt{g})(b_{qL}\mp b_{qR}) \pm (\frac{1}{\sqrt{g}}-\sqrt{g})(b^\dagger_{qL}\pm b^\dagger_{qR})\}$$

$$\hat{N}_+ \equiv \frac{1}{2}(\hat{N}_L - \hat{N}_R), \quad \hat{N}_- \equiv \frac{1}{2}(\hat{N}_L + \hat{N}_R)$$

$$\Phi_\pm(x) \equiv -\sum_{q>0}\frac{1}{\sqrt{n_q}}e^{-aq/2}[e^{-iqx}B_{q\pm} + e^{iqx}B^\dagger_{q\pm}]$$

$$= \frac{1}{\sqrt{8}}\{(\frac{1}{\sqrt{g}}+\sqrt{g})[\tilde{\phi}_L(x)\mp\tilde{\phi}_R(-x)] \pm (\frac{1}{\sqrt{g}}-\sqrt{g})[\tilde{\phi}_L(-x)\mp\tilde{\phi}_R(x)]\} \tag{161}$$

$$\rho_\pm(x) \equiv \partial\Phi_\pm(x) + \frac{2\pi}{L}\sqrt{2}g^{\pm 1/2}\hat{N}_\pm$$

B.3. **Tunneling density at site $x = 0$ without impurity.** In appendix B.2 we diagonalize the fermion interaction Hamiltonian, so that the Hamiltonian is free of interaction expressed in terms of transformed boson operators. We are interested in the tunneling density:

$$\rho_{dos}(\omega) \equiv \int_{-\infty}^{\infty}\frac{dt}{2\pi}e^{i\omega t}\langle G|\,\Psi_{phy}(t, x=0)\Psi^\dagger_{phy}(0, x=0)\,|G\rangle \tag{162}$$

In our case we set $x = 0$, from (144) and (146):

$$\Psi_{phy}(x=0)a^{-1/2}(F_L e^{-i\tilde{\phi}_L} + F_R e^{-i\tilde{\phi}_R})(x=0) \tag{163}$$

The left and right boson field can be expressed as the transformed boson field through (161):

$$\phi_{L/R}(x=0) = \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{g}}\Phi_-(x=0) \pm \sqrt{g}\Phi_+(x=0)) \tag{164}$$

Therefore, the fermion field can be completely expressed as boson field in free Hamiltonian:

(165) $$\Psi_{phy}(x=0) = a^{-1/2} e^{-\frac{i}{\sqrt{2g}}\Phi_-}(F_L e^{-i\sqrt{g/2}\Phi_+} + F_R e^{i\sqrt{g/2}\Phi_+})$$

Since $H = H_+ + H_-$, and $[H_+, H_-] = 0$, $\langle G| \Psi_{phy}(t, x=0)\Psi^\dagger_{phy}(0, x=0) |G\rangle = D_-(t)D_+(t)$. Where

(166)
$$D_-(t) \equiv \langle 0_-| e^{iH_-t}(e^{-\frac{i}{\sqrt{2g}}\Phi_-})e^{-iH_-t}(e^{\frac{i}{\sqrt{2g}}\Phi_-}) |0_-\rangle \quad \text{it only has } \Phi_- \text{ part}$$
(167)
$$D_+(t) \equiv a^{-1} \langle 0_+| e^{iH_+t}(F_L e^{-i\sqrt{g/2}\Phi_+} + F_R e^{i\sqrt{g/2}\Phi_+})e^{-iH_+t}(F_L^\dagger e^{i\sqrt{g/2}\Phi_+} + F_R^\dagger e^{-i\sqrt{g/2}\Phi_+}) |0_+\rangle$$
(168)
$$= D_{LL}(t) + D_{RR}(t) + D_{LR}(t) + D_{RL}(t)$$

Without impurity, $H-$ and H_+ are just free Hamiltonian, so that the evaluation up to a constant is:

(169) $$D_-(t) \sim e^{\frac{1}{2g}\langle 0_-|\Phi_-(t)\Phi_-(0)-\Phi_-(0)\Phi(0)|0_-\rangle} = (1 + ivt/a)^{-\frac{1}{2g}}$$

(170) $$D_+(t) \sim a^{-1}(1 + ivt/a)^{-\frac{g}{2}}$$

Therefore, the tunnelign density is proportional to:

$$\rho \sim \omega^{\nu-1}, \quad \nu = \frac{1}{2}(1/g + g)$$

For $g = 1$, the tunneling density is a constant and does not vanish at Fermi energy. But it is different for $g \neq 1$, the tunneling density vanishes at Fermi energy.